One can compare these spherical shells with analogous planar layered panels designed for the same conditions [4], which shows that the thickness and structure of a spherical shell tend to those for a planar panel as the radius increases; the two become virtually identical for $R_0 > 2$ m.

NOTATION

r, current radius; R_0, inside radius of sphere; R, outside radius; λ, a, ρ, thermal conductivity, thermal diffusivity, and density, respectively; T, temperature; τ, time; τ^*, period; ω, frequency of temperature fluctuations; α_1, α_2, heat-transfer coefficients.

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AN INVERSE STEFAN'S PROBLEM IN CASTING MULTICOMPONENT ALLOYS

Ya. F. Rutner

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A model is considered that incorporates the feature that melting (crystallization) occurs over a certain temperature range, and it is shown that the solution to the internal inverse problem is unique in one such formulation.

Determining the temperature pattern in a casting during crystallization is a nonlinear problem of the free-boundary class, in which part of the boundary is unspecified and must be determined when the differential equations are solved by the use of an additional boundary condition at that part. This relationship is readily derived from the heat-balance equation and is called Stefan's condition, which in the one-dimensional case takes the form

$$\left[\lambda_1 \frac{\partial}{\partial x} T_1(x,\ t) - \lambda_2 \frac{\partial}{\partial x} T_2(x,\ t)\right]\Bigg|_{x=s(t)} = r\gamma s'(t).$$

There are many papers on this topic, but in them it is either assumed that the crystallization occurs at a fixed temperature rather than over a certain range or that the treatment can be reduced to that.

We consider a schematic model for the phase-transition zone in casting a multicomponent alloy. During melting (cooling), a transitional layer is formed, which may be considered as a thermally active resistance.

Let the transitional layer have thickness $\delta > 0$, which may be fairly small. We denote by $R_0(x)$ the thermal-resistance density in the transitional layer, $R_0(x) = R'(x)$, while $r_0(x)$ denotes the density of the phase-transition latent heat, $r_0(x) = r'(x)$; we now apply Kirchoff's and Ohm's laws to the part $[x, x + dx]$ of the transition layer to get

$$\mathrm{dI}(x,\ t) = \gamma s'(t)\, r_0(x)\, \mathrm{dx}, \quad \mathrm{dT}(x,\ t) = I(x,\ t)\, R_0(x)\ \mathrm{dx},$$

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so

$$I(x,\ t)=\int\limits_{s(t)}^{x}\gamma s'(t)\,r_0(\xi)\,d\xi+C_1, \tag{1}$$

$$T(x,\ t)=\int\limits_{s(t)}^{x}I(\xi,\ t)\,R_0(\xi)\,d\xi+C_2, \tag{2}$$

where I(x, t) in (2) is defined in accordance with (1). The constants C_1 and C_2 are defined from the condition

$$I[s(t),\ t]=-\lambda_1\frac{\partial}{\partial x}T_1[s(t),\ t],\quad T[s(t),\ t]=T_s.$$

From (1) and (2) we get $C_1 = I[s(t),\ t]$, $C_2 = T_s$, which gives

$$I(x,\ t)=\gamma s'(t)\int\limits_{s(t)}^{x}r_0(\xi)\,d\xi-\lambda_1\frac{\partial}{\partial x}T_1[s(t),\ t], \tag{3}$$

$$T(x,\ t)=\int\limits_{s(t)}^{x}I(\xi,\ t)\,R_0(\xi)\,d\xi+T_s. \tag{4}$$

We write these results in more general form, which includes a model with lumped parameters:

$$I(x,\ t)=\gamma s'(t)\int\limits_{s(t)}^{x}dr(\xi)-\lambda_1\frac{\partial}{\partial x}T_1[s(t),\ t], \tag{5}$$

$$T(x,\ t)=\int\limits_{s(t)}^{x}I(\xi,\ t)\,dR(\xi)+T_s, \tag{6}$$

where by the integral we understand Lebesgue—Stieltjes one. However, with $x = s(t) + \delta$

$$I[s(t)+\delta,\ t]=-\lambda_2\frac{\partial}{\partial x}T_2[s(t)+\delta,\ t],$$

$$T[s(t)+\delta,\ t]=T_2[s(t)+\delta,\ t]=T_l.$$

Then (5) and (6) give

$$\lambda_1\frac{\partial}{\partial x}T_1[s(t),\ t]-\lambda_2\frac{\partial}{\partial x}T_2[s(t)+\delta,\ t]=\gamma s'(t)\int\limits_{s(t)}^{s(t)+\delta}dr(\xi), \tag{7}$$

$$T_l-T_s=\int\limits_{s(t)}^{s(t)+\delta}I(\xi,\ t)\,dR(\xi). \tag{8}$$

We consider the following uniform densities as an example:

$$R_0(x)=R_0=\frac{R}{\delta}=\text{const},\quad r_0(x)=r_0=\frac{r}{\delta}=\text{const}.$$

For this distribution we have

$$\lambda_1\frac{\partial}{\partial x}T_1[s(t),\ t]-\lambda_2\frac{\partial}{\partial x}T_2[s(t)+\delta,\ t]=\gamma s'(t)\,r, \tag{9}$$

$$I(x,\ t)=\gamma s'(t)\frac{r}{\delta}[x-s(t)]-\lambda_1\frac{\partial}{\partial x}T_1[s(t),\ t], \tag{10}$$

$$T_l-T_s=\frac{1}{2}\gamma s'(t)\,rR-\lambda_1 R\frac{\partial}{\partial x}T_1[s(t),\ t]. \tag{11}$$

If the process stabilizes for t → + ∞ and $\frac{\partial}{\partial x}T_1[s(t),\ t]$ tends to a constant value, then s'(t) tends to the corresponding constant value according to (11).

Therefore, if there are distributed parameters in the transitional layer, which in general occurs in all phase-transformation processes, the phase-transition boundary is not described by the square-root law, and the deviation from it will be the greater the more rapidly the process stabilizes under given conditions.

On considering a model with lumped parameters ($\delta = 0$), we find that the boundary condition system for the phase-transition region becomes

$$T_l-T_s=R_1\lambda_1\frac{\partial}{\partial x}T_1[s(t),\ t]+R_2\lambda_2\frac{\partial}{\partial x}T_2[s(t),\ t], \tag{12}$$

$$\lambda_1\frac{\partial}{\partial x}T_1[s(t),\ t]-\lambda_2\frac{\partial}{\partial x}T_2[s(t),\ t]=r\gamma s'(t), \tag{13}$$

$$T_1[s(t),\ t]=T_s, \tag{14}$$

$$T_2[s(t),\ t]=T_l, \tag{15}$$

where R_1 is the thermal resistance of the sublayer adjoining the solid phase and R_2 is that adjoining the liquid one.

These conditions differ substantially from those in the corresponding system of Stefan's conditions. The differences apply to (12), (14), and (15). In an idealized Stefan model, it is assumed that $R_1 = R_2 = R = 0$, so (12) becomes

$$T_l-T_s=0$$

when

$$T_l=T_s=T_{\text{mp}}.$$

We note that this general model corresponds to Academician Bochvar's viewpoint, according to which the solidification region is divided into two parts: the liquid–solid region, which includes the zone of microscopic liquid displacements, and the solid–liquid one, which includes the zones of local and microscopic liquid displacements.

This model is a refinement of the ideal Stefan model. If $\Delta T = T_l - T_s$ is small by comparison with T_s, the errors in estimating the phase-transition boundary and the temperature distribution will be fairly small. On the other hand, these errors increase substantially with ΔT.

We consider the internal inverse problem: determining the boundary condition at the immobile boundary x = 0 to provide a given law x = s(t) for the displacement of the phase boundary by the use of additional information on the solution at internal points.

We substitute this inverse problem into the model with lumped parameters:

$$\frac{\partial T_1}{\partial t}=a_1\frac{\partial^2 T_1}{\partial x^2},\quad t>0,\ -\infty<x<s(t),\quad s(0)=0, \tag{16}$$

$$\frac{\partial T_2}{\partial t}=a_2\frac{\partial^2 T_2}{\partial x^2},\quad t>0,\quad s(t)<x<0, \tag{17}$$

$$T_1(x,\ 0)=T_s,\ -\infty<x<0, \tag{18}$$

$$T_1(-\infty,\ t)=T_s,\quad t\geqslant 0, \tag{19}$$

$$T_1[s(t),\ t]=T_s,\quad t>0, \tag{20}$$
$$T_2[s(t),\ t]=T_l,\quad t>0, \tag{21}$$

$$T_l-T_s=R_1\lambda_1\frac{\partial}{\partial x}T_1[s(t),\ t]+R_2\lambda_2\frac{\partial}{\partial x}T_2[s(t),\ t], \tag{22}$$

$$\lambda_1\frac{\partial}{\partial x}T_1[s(t),\ t]-\lambda_2\frac{\partial}{\partial x}T_2[s(t),\ t]=r\gamma s'(t), \tag{23}$$

$$T_2(x_0,\ t)=\varphi(t). \tag{24}$$

From this system of equations, we have to determine $T_1(x,\ t)$, $T_2(x,\ t)$, and in particular $T_2(0,\ t) = f(t)$ with a known law $x = s(t)$ for the displacement of the phase boundary.

Theorem. The phase-boundary displacement law can only be uniform. This law is provided by the function

$$f(t)=T_l\exp\left(\frac{v^2t}{a_2}\right)+\left(\frac{r\gamma a_2}{\lambda_2}-T_l\right)\left[\exp\left(\frac{v^2t}{a_2}\right)-1\right] \tag{25}$$

for

$$\varphi(t)=T_l\exp\left[\frac{v}{a_2}(x_0+vt)\right]+\left(\frac{r\gamma a_2}{\lambda_2}-T_l\right)\left\{\exp\left[\frac{v}{a_2}(x_0+vt)\right]-1\right\}, \tag{26}$$

where v is the phase boundary displacement rate.

Proof. 1°. On considering the boundary-value problem of (16) and (18)-(20), we see that it has a unique solution by virtue of certain theorems. As $T_1(x,\ t) \equiv T_S$ is the solution to this problem, as can be verified by direct substitution, this solution is unique. Then we arrive at an inverse formulation in the one-phase form (the subscript 2 is omitted below):

$$\frac{\partial T}{\partial t}=a\frac{\partial^2 T}{\partial x^2},\ t>0,\ s(t)<x<0, \tag{27}$$

$$T[s(t),\ t]=T_l,\quad s(0)=0, \tag{28}$$

$$T_l-T_s=R\lambda\frac{\partial}{\partial x}T[s(t),\ t], \tag{29}$$

$$-\lambda\frac{\partial}{\partial x}T[s(t),\ t]=r\gamma s'(t), \tag{30}$$

$$T(x_0,\ t)=\varphi(t). \tag{31}$$

2°. On comparing (29) and (30) we get that

$$-r\gamma s'(t)=\frac{T_l-T_s}{R},$$

when

$$s'(t)=-\frac{T_l-T_s}{Rr\gamma},$$

by virtue of which $s(t) = -vt + C$, where

$$v=\frac{T_l-T_s}{Rr\gamma}=\text{const}, \tag{32}$$

and C is an arbitrary constant.

As $s(0) = 0$, we have $C = 0$, and therefore $s(t) = -vt$. From (32) with a known v we readily get the thermal resistance:

$$R = \frac{T_l - T_s}{r\gamma v}.$$

3°. We than get a noncharacteristic Cauchy problem:

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2},\quad t > 0,\quad -vt < x < 0,\quad T(-vt,\ t) = T_l,\ t > 0,$$

$$\frac{\partial}{\partial x} T(-vt,\ t) = \frac{r\gamma v}{\lambda}$$

with data on the straight line $x = -vt$. It is known [1] that if the solution to this problem exists, it is unique.

4°. We now use some known results [2] on the noncharacteristic Cauchy problem for the heat-conduction equation to get the solution as

$$T(x,\ t) = T_l \exp\left[\frac{v}{a}(x + vt)\right] + \left(\frac{r\gamma a}{\lambda} - T_l\right)\left\{\exp\left[\frac{v}{a}(x + vt)\right] - 1\right\}, \tag{33}$$

whence we derive the function $T(0,\ t) = f(t)$ as

$$f(t) = T_l \exp\left(\frac{v^2 t}{a}\right) + \left(\frac{r\gamma a}{\lambda} - T_l\right)\left[\exp\left(\frac{v^2 t}{a}\right) - 1\right].$$

5°. Let the phase boundary displacement rate be an unknown quantity. From (33) we get

$$\varphi(t) = T_l \exp\left[\frac{v}{a}(x_0 + vt)\right] + \left(\frac{r\gamma a}{\lambda} - T_l\right)\left\{\exp\left[\frac{v}{a}(x_0 + vt)\right] - 1\right\},$$

from which v can be determined.

NOTATION

λ, thermal conductivity; a, thermal diffusivity; γ, density, $\gamma_1 = \gamma_2 = \gamma$; t, time; x, spatial coordinate; $x = s(t)$, phase boundary; r, latent heat of melting; $T(x,\ t)$, temperature; T_l, liquidus temperature; T_s, solidus temperature. Subscripts: $i = 1$, solid phase; $i = 2$, liquid phase.

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